Predicting Navigation Fix Accuracy :
A Realistic Alternative to Over-Optimistic DOP Values

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Abstract

DOP (Dilution of Precision) "defines" how good a position fix is. If only it was so simple. Like too many statistical metrics, DOP depends more on the assumptions underpinning the analysis than on the data from which the fix is computed.

It is well known in the surveying community that symmetric geometries are highly beneficial where accurate, unbiased position solutions are desired. However, where symmetry is lacking, such as at the edge of a tracking range, biased fixes are likely to result. Significantly, the usual DOP analysis and metrics do not identify this performance shortfall and are seriously flawed.

As tracking range deployments become more frequent, performance analysis techniques and tools are increasingly important to mission success. The limitations of the DOP paradigm and an alternative metric which encompasses intuition and is in operational use will be described with comparative examples presented.

Introduction

The focus of this paper is on defining insightful analysis techniques for underwater tracking range design. The techniques are targeted at relatively complex problems commonly involving tens of sensors, propagation constraints and bottom bathymetry. Self evidently, the techniques are applicable to other, similar sensor network optimisation problems.

This paper is, of course, in the Practical Results session. However, for the usual reasons of confidentiality, it must eschew the use of client's data sets. Additionally, the techniques considered are analytic techniques for use *a priori* generally before a range has been deployed and objective performance data obtained.

Nonetheless to quote Kurt Lewin [1] *"There is nothing so practical as a good theory"*. We consider, based on practical experience, that the techniques presented are an example of a good theory, which appropriately applied, benefits the art of navigation. The examples presented are both synthetic and, in the case of the European Loran stations, wholly practical.

Emeritus Solutions Ltd is of course experienced in the analysis and synthesis of underwater tracking range layouts, with a profound understanding of the applicable mathematics, to underpin their analysis tools. Emeritus Solutions Ltd can provide (bespoke) tools for requirements ranging from range layout analysis and synthesis, through the computation of tracking solutions and sensor boxing, to the detailed analysis of range and environmental data. Equally, Emeritus Solutions Ltd can draw on a wide range of underwater acoustic, digital signal processing, computing and electronics competencies. These offerings can be provided on a turnkey, service or consultancy basis.

This paper first identifies key aspects of the underwater positioning problem, which both differentiate it from the GNSS (satellite navigation) problem and make the computation of accurate positions a challenging task. The trilateration problem is then described "In a Nutshell" and a key Gdop bound result is presented; all with the minimum of mathematics. Performance analyses using the DOP paradigm are then presented for a range of specimen tracking range geometries. Our support metric technique is then presented to illustrate the greater discrimination achieved both on specimen geometries and in a real world example – the European Loran chains. Our conclusions are then plainly stated.

Underwater Positioning – Key Differences from GNSS

Almost all of the trilateration literature [see, for example, 2 – 23] addresses radio navigation; from early works on Loran (and Gee), through the early days of GPS, to contemporary GNSS and mobile phone tracking. Consequently, the mode of operation and important issues for underwater positioning can differ appreciably from those of interest in radio navigation. The following paragraphs identify some of the key differences between underwater positioning and GNSS.

Clock performance is typically a significant issue for underwater positioning systems. Somewhere in the system sub-Rubidium performance will commonly "be a problem". At a minimum this will require the system to estimate emission times and work from pseudo, rather than absolute, ranges. One consequence, is that an additional pseudo range measurement is required – which in turn requires a higher sensor density. Another, more significant, consequence is that position fixing outwith the sensor array is very imprecise – an issue which is perhaps less problematic for GNSS systems.

Propagation velocity is invariably an issue. The speed of sound in water invariably varies with depth, time of year, time of day and location. At a minimum it must be directly measured, and such is its variation with depth that its harmonic mean must be used. The most radical difference from radio navigation is that the uncertainty of propagation velocity can be sufficiently large to significantly affect position solutions. Again, working within a (preferably symmetric) sensor array is the most effective mitigation.

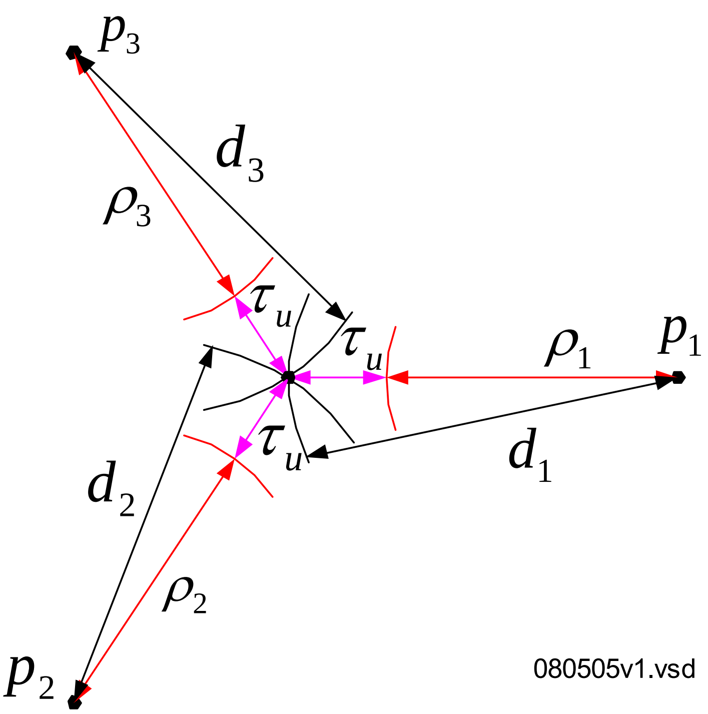
Unfortunately, many other difficulties are attributable to propagation velocity – specifically to its variation with depth. GNSS requires mitigation of ionosperic and tropospheric propagation effects at low elevation angles, and has the option of masking out low elevation satellites. As most channels of interest are longer than they are deep, underwater positioning is typically performed at low elevation angles. Consequently, shadowing and variable path lengths due to refraction are the norm, both imposing (sometimes absolute) constraints and requiring ingenuity in system design.

An important consequence of the "shallow" environment in which underwater positioning is performed is that in general depth is measured and if necessary telemetered rather than estimated by trilateration. A positive aspect of this circumstance is that the positioning problem becomes two dimensional, and its solution requires one fewer pseudo range measurement. There are of course exceptions, such as sensor boxing, where the slant range is only a few times the water depth.

Finally, we shall simply mention some of the "second order" issues which if not appropriately mitigated can comprehensively "ruin your day", for example: anisotropic propagation velocity due to currents, multipath and surface reverberation.

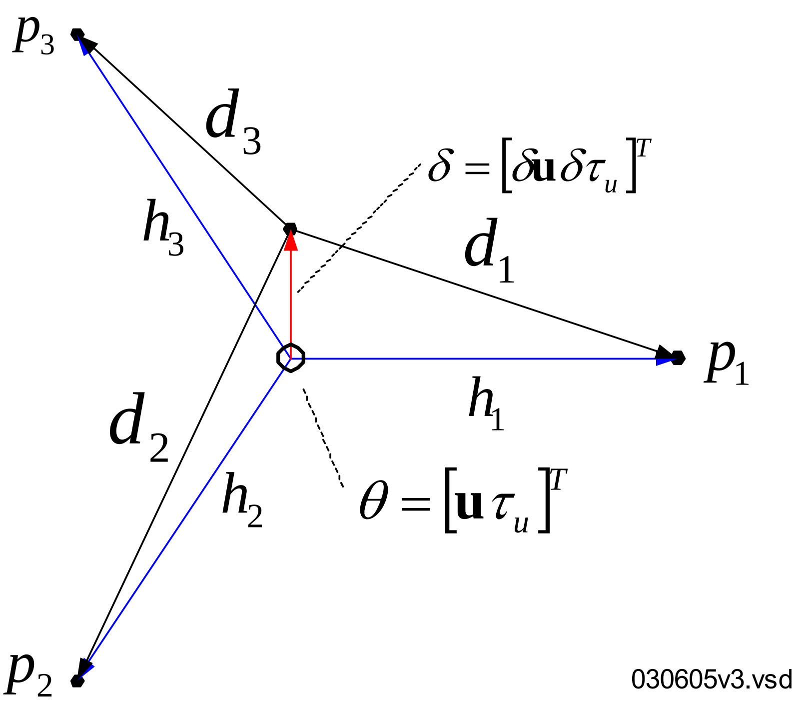
Trilateration - In a Nutshell

The preceding section indicated that enclosed geometries with significant symmetry are, if not always possible, certainly desirable. Therefore, as these are in any case the simplest trilateration geometries to analyse, we shall use them to set out the trilateration stall. General theories and analytic tools are of course available for arbitrary geometries. However, simple geometries yield both useful rules of thumb and valuable insights.



Trilateration Problem in 2-D

The preceding figure depicts the trilateration problem for three sensors deployed in an equilateral triangle. This configuration uses the minimum number of sensors for estimating 2-D position (x & y) and the user clock (epoch of emission). Further regular n-gons squares, pentagons, hexagons, etc can of course implement over-determined solutions. The observables are the pseudo ranges ρ_i which differ from the true ranges d_i by the user clock offset t_u and additive noise terms n_i.



Trilateration Iteration in 2-D

The preceding figure depicts the solution framework. The sensor geometry is described by the direction cosines $\underline{h}_i$, which are aligned with the user – sensor vectors h_i. The estimated user position is given by the column vector θ with elements x, y and t_u. Under the standard assumptions of independent, equal, zero mean, Gaussian noise processes the estimator for the user state and the covariance of the estimate may be stated as

$$\varepsilon = \rho - \underline{H}.(P - \theta)$$

$$\delta\theta = \left(\underline{H}^T \underline{H}\right)^{-1} \underline{H}^T \varepsilon$$

$$\mathrm{cov}(\theta) = \sigma_r^2 \left(\underline{H}^T \underline{H}\right)^{-1} = \begin{bmatrix} \sigma_{xx}^2 & . & . & . \\ . & \sigma_{yy}^2 & . & . \\ . & . & \sigma_{zz}^2 & . \\ . & . & . & \sigma_{tt}^2 \end{bmatrix}$$

In brief, the ε is the vector of innovations or residuals, a mixture of (small) offsets and noise which the iterative estimator (for θ) will minimise. θ is the estimate of position and user clock offset x, y, t_u and potentially z which constitutes the solution to the trilateration problem. δθ is an increment in the iterative computation of θ. Cov(θ) is the covariance of the position estimates obtained for a specified sensor geometry $\underline{H}$.

DOP can now be defined, for the simple case where the ranging errors have equal variances, as the ratio of cov(θ) to σ_r^2. In particular, the diagonal terms and their aggregates provide expressions for Xdop, Ydop, etc and for Hdop, Pdop and Gdop. These DOP values provide an indication of estimator performance for a given user / sensor geometry, and are the standard method of assessing system performance [see, for example, 12, 13, 21 & 22].

$$Xdop = \sqrt{(\underline{H}^T \underline{H})^{-1}{}_{xx}} = \sqrt{\sigma_{xx}^2 / \sigma_r^2}\,;etc \ \ for\ Ydop\ et\,al$$

$$Hdop = \sqrt{(\underline{H}^T \underline{H})^{-1}{}_{xx} + (\underline{H}^T \underline{H})^{-1}{}_{yy}} = \sqrt{(\sigma_{xx}^2 + \sigma_{yy}^2) / \sigma_r^2}$$

$$Pdop = \sqrt{(\underline{H}^T \underline{H})^{-1}{}_{xx} + (\underline{H}^T \underline{H})^{-1}{}_{yy} + (\underline{H}^T \underline{H})^{-1}{}_{zz}} = \ldots$$

$$Gdop = \sqrt{(\underline{H}^T \underline{H})^{-1}{}_{xx} + (\underline{H}^T \underline{H})^{-1}{}_{yy} + (\underline{H}^T \underline{H})^{-1}{}_{zz} + (\underline{H}^T \underline{H})^{-1}{}_{tt}} = \ldots$$

Obviously, Gdop the geometric dilution of precision is a multiplicative factor relating the uncertainty of the position estimate to the user range error σ_r^2. Evidently, a low value of Dop is desirable – just as a good crossing angle for two lines of position is 90°. A good way of obtaining a feel for Gdop values is to examine them in 2-D space for regular n-gons. From this we can obtain an indication of minimum values and their location, the typical range of values, and for the regular n-gons which are the building blocks of most tracking ranges some insight into the interaction between sensor geometry and DOP.

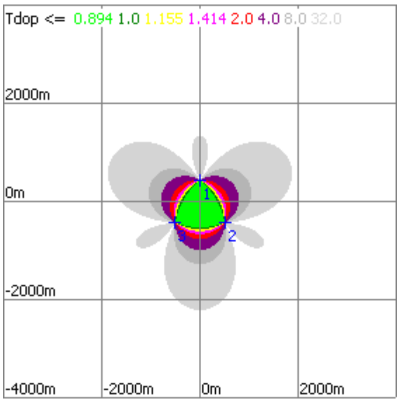
Levanon [23] derived a simple expression for Gdop at the centre of a regular polygon

$$Gdop = 2 \big/ \sqrt{N}$$

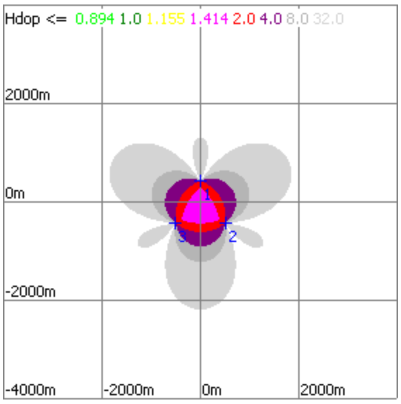
evidently Gdop reduces as the square root of N, that is to halve Gdop the number of sensors must be quadrupled. This square root relationship between the number of (independent) measurements and the accuracy of the estimate is of course unsurprising. Nonetheless, it provides a strong indication of the general relationship between the number of sensors and accuracy. Additionally, it can be shown that regular polygons minimise dop.

GDOP Analysis Examples

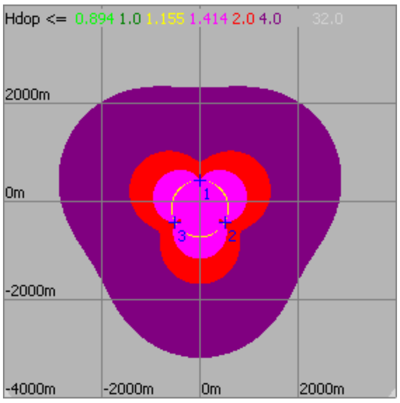
It does of course remain to partition Gdop between Pdop, the (xyz) position dilution of precision, and Tdop, the time dilution of precision. In the 2-D case one can work with either pseudo ranges, where the clock offset / emission time is estimated, or with absolute ranges where the clock offset / emission time is taken as known. As the following plots show, the use of an absolute range system promises improved coverage, especially outside the sensor array.



Tdop – pseudo ranges



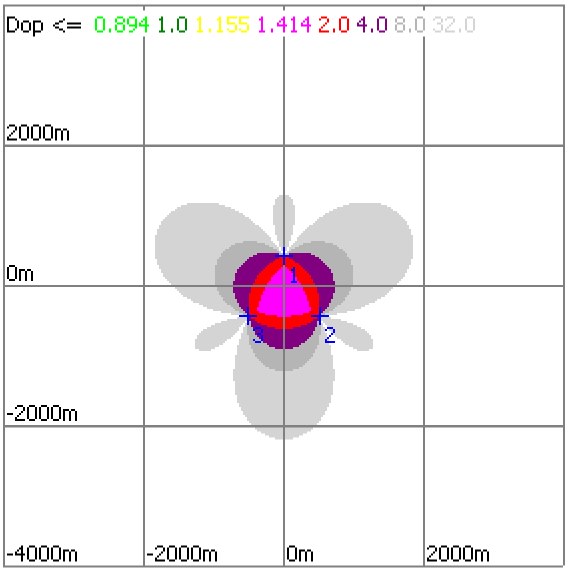
Hdop – pseudo ranges



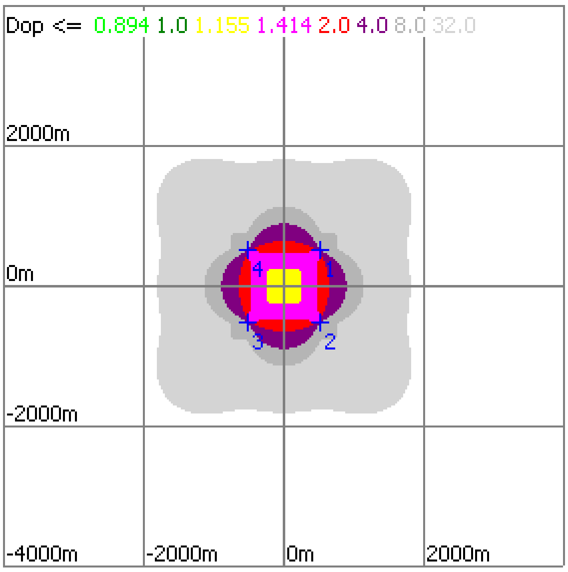
Hdop - absolute ranges

Important points to note are that in the pseudo range case Hdop and to a greater degree Tdop are only low inside the triangular sensor array. Another important point is that the improved coverage of the absolute range (synchronous) case is predicated on synchronisation; it can readily be shown that if this assumption is violated the position fixes are totally compromised. The benefits of working inside the sensor array, preferably with a balanced sensor geometry are evident.

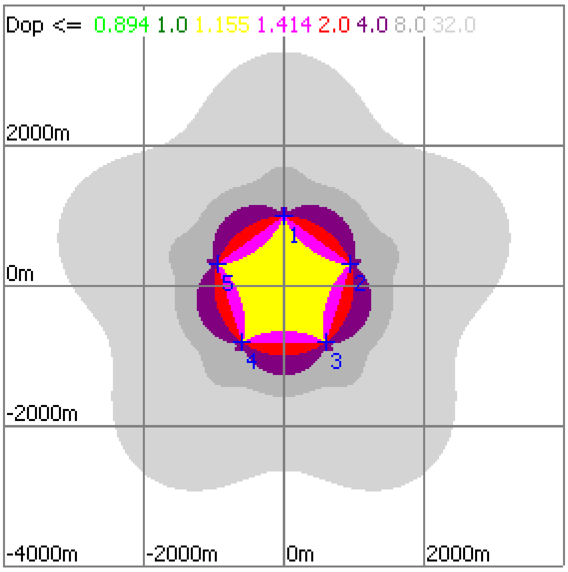
Our illustration of the trilateration performance of n-gons is completed by the following Gdop plots for a triangle, a square, a pentagon and a centre point hexagon. These figures include all the regular polygons commonly used as elements of tracking ranges. The centre-point hexagon is of course a tiling of equilateral triangles, the tiling commonly used for tracking ranges. Regular tilings of the plane are of course only available for triangles, squares and hexagons. Additionally, analysis of normalised coverage (area / sensors) for a regular array under the constraint of a maximum inter sensor spacing clearly demonstrates the optimality for many requirements of a regular tiling of equilateral triangles.



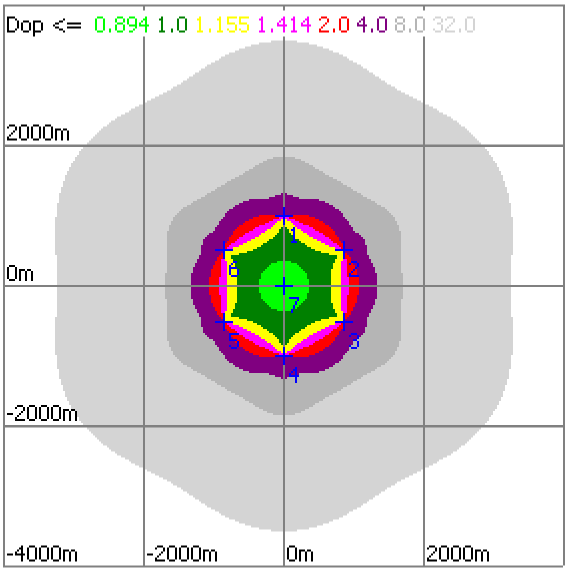
Gdop - triangle



Gdop - square



Gdop – pentagon



Gdop – centre-point hexagon

Again, the location of good coverage is within the sensor array especially where the sensor geometry is balanced. Nonetheless it is evident that Dop has limitations as an analytic metric. Firstly, it does not implicitly identify the weakness of working outside the sensor array. Secondly, it is difficult to normalise for an effective visual appreciation of support for tracking at all locations.

An Alternative Metric to DOP

The simplest way to motivate our alternative metric is to recast the covariance expressions firstly in terms of determinants and then in terms of the lengths, areas and volumes defined by the direction cosines of the $\underline{H}$ matrix.

The expression for DOP can be reformulated in terms of the trace of the matrix $(\underline{H}^T\underline{H})^{-1}$

$$Gdop = \sqrt{(\underline{H}^T\underline{H})^{-1}{}_{xx} + (\underline{H}^T\underline{H})^{-1}{}_{yy} + (\underline{H}^T\underline{H})^{-1}{}_{zz} + (\underline{H}^T\underline{H})^{-1}{}_{tt}} = \sqrt{tr(\underline{H}^T\underline{H})^{-1}}$$

Now, if $\underline{H}^{-1}$ exists the trace can be rewritten in terms of $\underline{H}$'s adjoint matrix and determinant

$$tr(\underline{H}^T\underline{H})^{-1} = \left(1\big/\left|\underline{H}\right|^2\right)\sum_{ij}(h'_{ij})^2$$

where $\underline{h}'_{ij}$ is the ij th element of $\underline{H}$'s adjoint matrix.

As the denominator dominates this formulation for Gdop (we shall offer insight as to why later) maximising the discriminant $|\underline{H}|$ should maximise Gdop. Now, it is well known, that that the determinants of real n x n matrices are equivalent to volumes in n-dimensional Euclidian space; see e.g. Birkhoff and Mac Lane [24]. In particular for a 2-D problem employing 3 sensors

$$|\underline{H}| = \begin{vmatrix} \underline{h}_1^x & \underline{h}_1^y & 1 \\ \underline{h}_2^x & \underline{h}_2^y & 1 \\ \underline{h}_3^x & \underline{h}_3^y & 1 \end{vmatrix} = \begin{vmatrix} \underline{h}_2^x - \underline{h}_1^x & \underline{h}_2^y - \underline{h}_1^y \\ \underline{h}_3^x - \underline{h}_1^x & \underline{h}_3^y - \underline{h}_1^y \end{vmatrix}$$

In this example the area equivalent to the determinant is that enclosed by the three direction cosines pointing towards the sensors. The 1's in the 3 x 3 matrix are of course the terms associated with the user epoch. The notation $\underline{h}^x$ and $\underline{h}^y$ indicates the x and y components of the direction cosines.

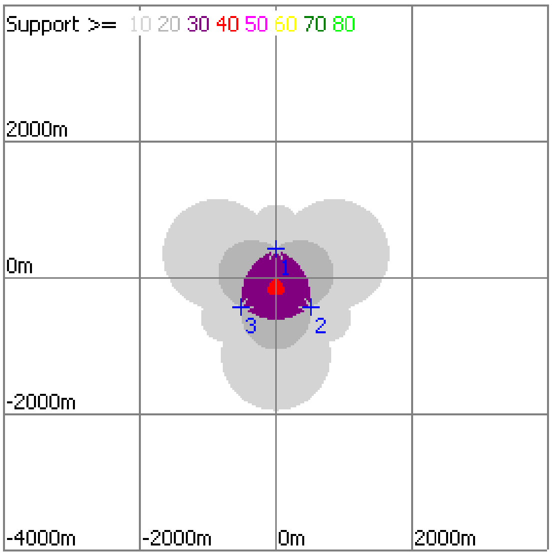
Papers by Phillips [19] and Massatt and Rudnick [20] comprehensively developed the expression of DOP in terms of lengths, areas and volumes. For example, given perfect knowledge of the z coordinates, the expression for Hdop in the 3 sensor case is

$$Hdop = \sqrt{\sum_{i=1}^{3} l_{xy_i}^2} \bigg/ 2b_{xy}$$

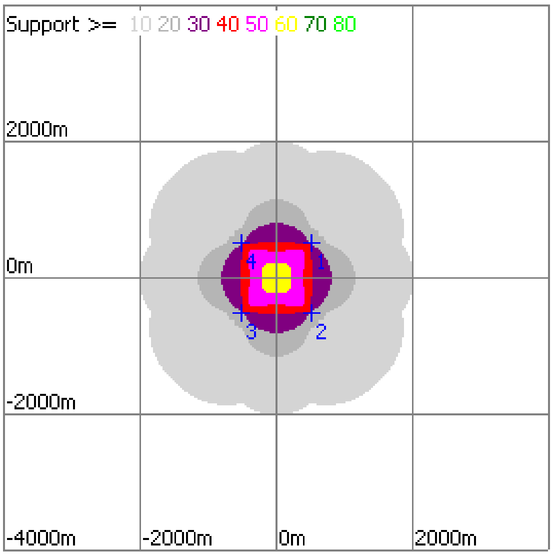
where l_{xyi} is the length of the projection on the xy plane of the side opposite the i^{th} direction cosine and b_{xy} is the area of the projection of the convex hull of the direction cosines onto the xy plane.

From this formula for Hdop we can see that the numerator, which is the root sum of squares of the lengths of the sides of the triangle, will vary much less than denominator. Consequently, as was observed earlier in this section, maximisation of the area b_{xy} or equivalently |$\underline{H}$| effectively minimises DOP.

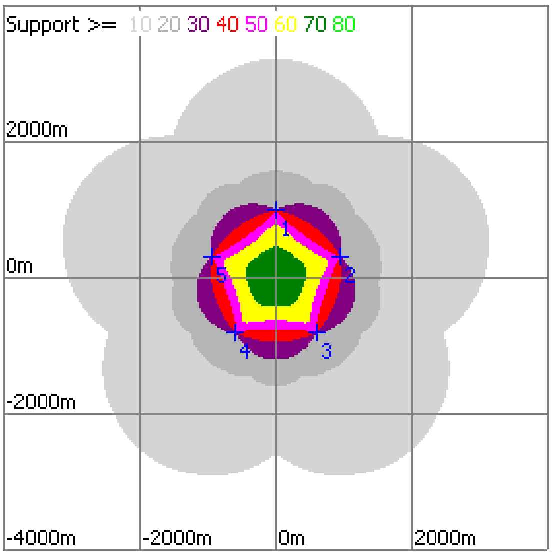
The following plots of normalised area demonstrate the sensitivity of the alternate metric to the support provided by a given sensor / user configuration for the usual sensor arrays: equilateral triangle, square, pentagon and centre point hexagon.



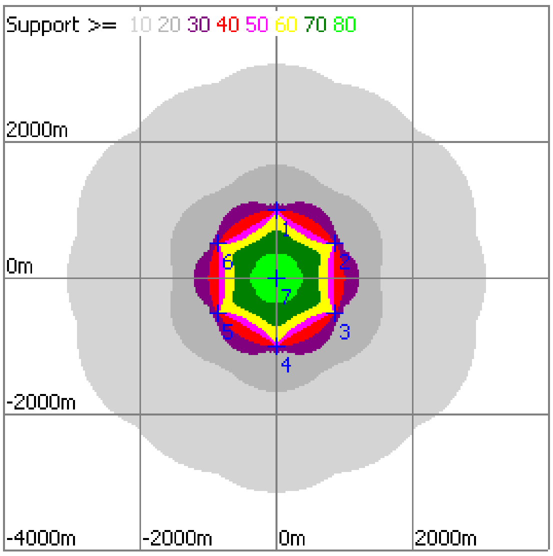
Support – equilateral triangle



Support - square



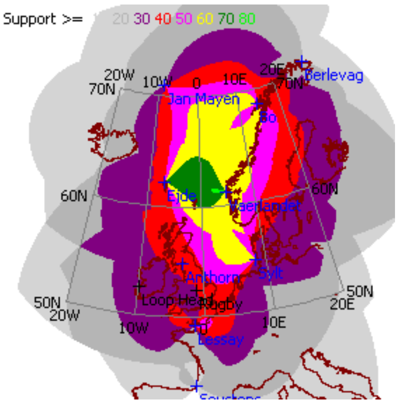
Support - pentagon



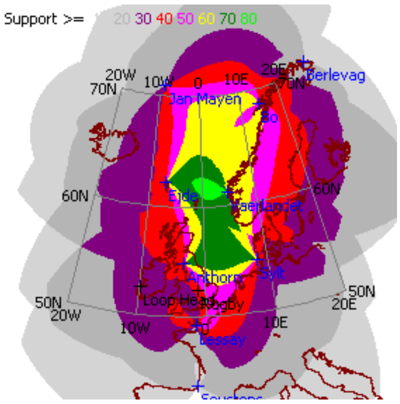
Support – centre-point hexagon

Self evidently the support metric provides a strong indication of both good and poor positioning geometries. The low values for support towards the edges and outside the sensor array do of course concur with the practitioner's expectation and experience.

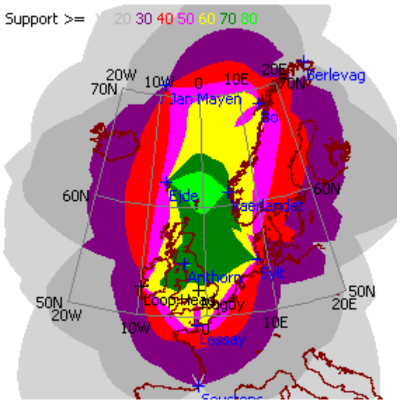
Most importantly, the support metric can meaningfully address real world problems with arbitrary sensor locations. As an example we present an analysis of several of configurations of the European Loran chain(s).



Baseline



+ Anthorn



+Anthorn + Loop Head

The baseline configuration comprises the "French" and "Norwegian" chains; Soustons, Lessay & Sylt and Værlandet, Ejde, Jan Mayen, Bø & Berlevåg respectively. The projection is UTM zone 31 (3° E) and the size of the grid squares 1000 km. The improved coverage with the addition of the UK station at Anthorn is evident. The additional improvement available were the Loop Head station operational is also plain. A range cut off of 2000 km has been applied to the analysis – this is broadly the prudent navigator's limit of dependable ground wave coverage. Equally, the prudent navigator would probably consider the coverage adequate for general navigation to the 50% or 40% support boundary, and beyond that for carefully observed and qualified fixes.

Conclusions

This paper has described an insightful analysis technique for underwater tracking range design. The support metric has been demonstrated using a mixture of examples, and its strong agreement with the standard rules of thumb demonstrated. The support metric is of course ideally suited to problems involving tens of sensors, propagation constraints and bottom bathymetry where detailed, analytic modelling is essential. The applicability of the support metric and related techniques to Loran coverage analysis has been demonstrated. The technique is undoubtedly applicable to many other sensor network analysis and optimisation problems.

We consider, based on practical experience, that the techniques presented are an example of a good theory, which appropriately applied, benefits the art of navigation. Once again to quote Kurt Lewin [1] *"There is nothing so practical as a good theory"*.

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Concluding Remarks

The author acknowledges many useful interactions with co-workers.

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